Integrality Gaps and Approximation Algorithms for Dispersers and Bipartite Expanders

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Presented by James Hong

Motivation

Vertex expansion in bipartite graphs:

$$G = ([N], [M], E)$$

Where maximal degree is

- **D** for vertices in [N] (*left degree*)
- **d** for vertices in [M] (*right degree*)

Interested in the size of neighbor sets:

- If $S \subseteq [N]$ or $S \subseteq [M]$, then

$$\Gamma(S) = \{j \mid \exists i \in S.\, (i, j) \in E\}$$

Definitions

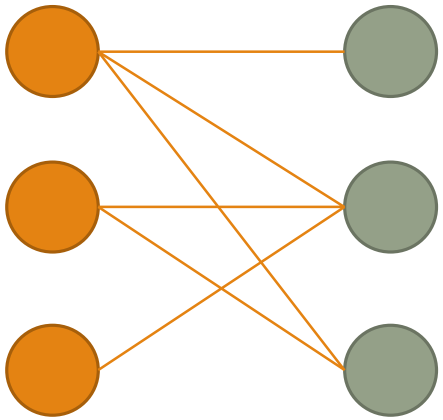
A bipartite graph $G = ([N], [M], E)$ is a

1. **(k,s)-disperser** if for any subset $S \subseteq [N]$ of size k,

$$|\Gamma(S)| \geq s$$

Example: Disperser

The graph below is a (2,2)-disperser. *(Also a (3,3) and (1,1)-disperser)*



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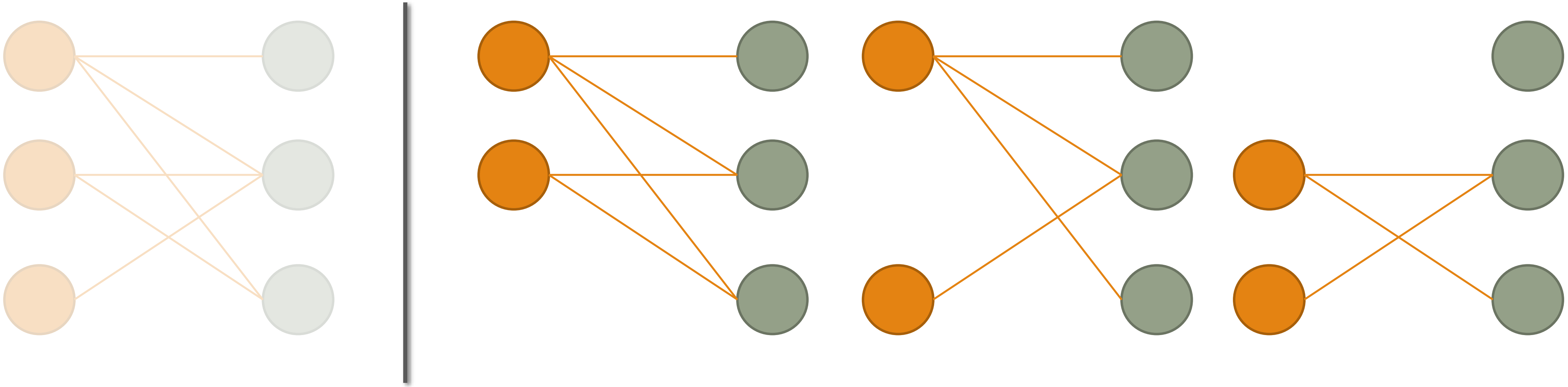

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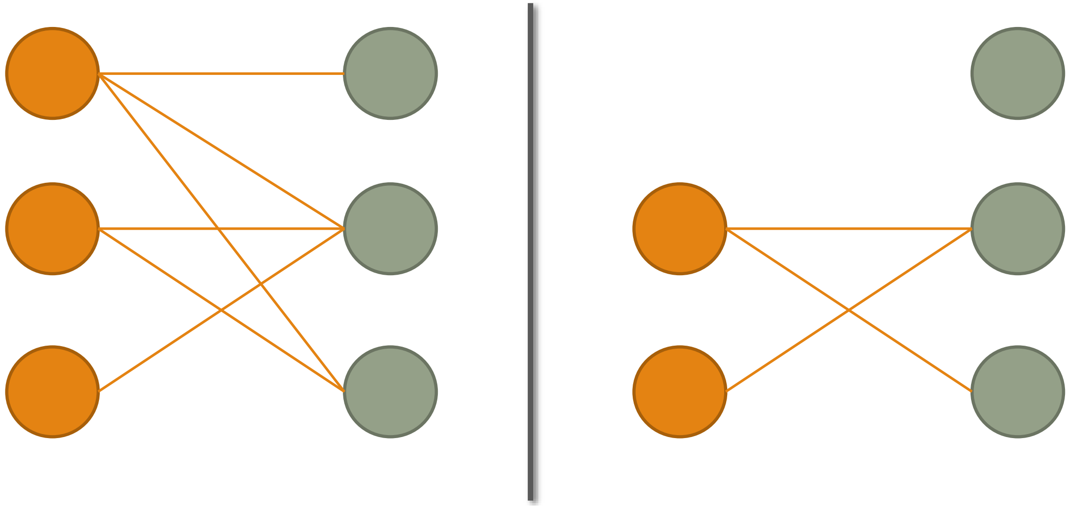
3. **(≤K,a)-expander** if for all $k \leq K$, G is a (k,a)-expander

Example: Expander

The graph below is a (2,1)-expander *(Also a (≤2,1)-expander)*



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$$k = \rho N, s = (1 - \delta)M$$

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Background & Applications

Dispersers

- Non-trivial derandomization results

MAX-Clique, deterministic amplification, oblivious routing

Expanders

- Studying pseudorandomness

expander codes and randomness extractors

Random graphs make good dispersers and expanders

W.h.p, for sufficiently large N and left degree $D = \Theta_{\alpha,\delta}(log\, N)$,

a random biprartite graph $G = ([N],[M],E)$ is a $(N^{\alpha},(1-\delta)M)$-disperser.

Paper Overview

1. SDP relaxation for vertex expansion
2. Proof of limits of the Lasserre hierarchy for
    - Distinguishing certain classes of dispersers and expanders
3. A poly-time approximation algorithm for finding a ρN sized subset with the smallest neighbor set
4. Hardness result based on *SSE* for the *disperser problem*

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Generally, the proofs assume that the graph is either d-regular (right) or D-regular (left) depending on the proof. There is the assumption that M or N are sufficiently large so that we can argue that the graphs are good dispersers or expanders.

Integer Program for Vertex Expansion

Specifically for ρN subsets of $[N]$

- Let x_i indicate inclusion/exclusion

$$\min \sum_{j=1}^{M} \vee_{i \in \Gamma(j)} x_i$$

subject to

$$\sum_{i=1}^{N} x_i \geq \rho N$$
$$\forall i \in [N].\ x_i \in \{0,1\}$$

- $\vee_{i \in \Gamma(j)} x_i$ is a constraint from $[M]$, the goal is to minimize the number of constraints satisfied.

Integer Program for Vertex Expansion

The objective becomes a CSP.

$$\min \sum_{j=1}^{M} \vee_{i\in\Gamma(j)} x_i = \min \sum_{j=1}^{M} 1 - 1_{\wedge i\in\Gamma(j), x_i=0} = M + \max \sum_{j=1}^{M} 1_{\wedge i\in\Gamma(j), x_i=0}$$

Highlights:

- Apply $\Omega(N)$ rounds of Lasserre
- The author derives an upper and lower bound on vertex expansion when $\rho = 1 - 1/q$ where q is the prime order of a finite field F_q (List-CSP).
- Generalizes the bounds for any ρ. [Solving several CSP and SDPs.]

Integrality Gap for the **Disperser** Problem

Theorem 1.1.

For $\alpha \in (0,1)$ and any $\delta \in (0,1)$, the $N^{\Omega(n)}$-level Lasserre hierarchy cannot distinguish whether G, a random bipartite graph with left degree $D = O(\log n)$

1. G is an $(N^{\alpha}, (1-\delta)M)$-disperser
2. G is not an $(N^{1-\alpha}, \delta M)$-disperser

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True w.h.p for a random graph

1. G is an $(N^{\alpha}, (1-\delta)M)$-disperser
2. G is not an $(N^{1-\alpha}, \delta M)$-disperser

SDP objective after $\Omega(N)$ levels of Lasserre for obtaining δM distinct neighbors is at least $N^{1-\alpha}$

Integrality Gap for the **Disperser** Problem

Theorem 1.2.

For any $\rho > 0$ there exist infinitely many d such that the $\Omega(N)$-level Lasserre hierarchy cannot distinguish, for a random bipartite graph G with right degree d, whether

1. G is an $(pN, \left(1-(1-\rho)^d\right)M)$-disperser
2. G is not an $(pN, (1 - C_0 \cdot \frac{1-\rho}{\rho d+1-\rho}))$-disperser for an universal constant $C_0 > 0.1$

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SDP objective after $\Omega(N)$ levels of Lasserre is at most this

Integrality Gap for the **Expander** Problem

Theorem 1.4.

For any $\epsilon > 0$ and $\epsilon' < \frac{e^{-2\epsilon}-(1-2\epsilon)}{2\epsilon}$, there exist ρ and D such that $\Omega(N)$-level Lasserre hierarchy cannot distinguish, for a bipartite graph G with left degree D, whether

1. G is an $(pN, (1-\epsilon')D)$-expander
2. G is not an $(pN, (1-\epsilon)D)$-expander

Integrality Gap for the Expander Problem

Theorem 1.4.

For any $\epsilon > 0$ and $\epsilon' < \frac{e^{-2\epsilon}-(1-2\epsilon)}{2}$, there exist ρ and D such that

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True w.h.p for a random graph

1. G is an $(pN, (1-\epsilon')D)$-expander
2. G is not an $(pN, (1-\epsilon)D)$-expander

SDP objective after $\Omega(N)$ levels of Lasserre is at most $(1-\epsilon)D \cdot \rho N$

Integrality Gap for the **Expander** Problem

Theorem 1.4.

For any $\epsilon > 0$ and $\epsilon' < \frac{e^{-2\epsilon}-(1-2\epsilon)}{2\epsilon}$, there exist ρ and D such that $\Omega(N)$-level Lasserre hierarchy cannot distinguish, for a bipartite graph G with left degree D, whether

1. G is an $(\boldsymbol{pN}, \mathbf{0.6322}\boldsymbol{D})$-expander
2. G is not an $(\boldsymbol{pN}, \mathbf{0.499}\boldsymbol{D}))$-expander

Smallest Neighbor Set Approximation Algorithm

Theorem 4.1.

Suppose the following:

- G has right degree d
- $(1-\Delta)M$ is the smallest neighbor set over ρN subsets of $[N]$

There is a polynomial time algorithm that outputs $T \subseteq [N]$, $|T| = \rho N$ and

$$\Gamma(T) \leq \left(1 - \Omega\left(\frac{\min\{\left(\frac{\rho}{1-\rho}\right)^2, 1\}}{\log d} \cdot d(1-\rho)^d \cdot \Delta\right)\right) M$$

Smallest Neighbor Set Approximation Algorithm

Algorithm approach:

1. Solve the SDP [Right] to maximize the number of unconnected vertices to T

$$\max \sum_{j \in [M]} \| \frac{1}{d} \sum_{i \in \Gamma(j)} \vec{v}_i \|_2^2$$

$$\text{Subject to} \quad \langle \vec{v}_i, \vec{v}_i \rangle \leq 1$$

$$\sum_{i=1}^{n} \vec{v}_i = \vec{0}$$

Smallest Neighbor Set Approximation Algorithm

Algorithm approach:

1. Solve the SDP [Right] to maximize the number of unconnected vertices to T
2. Round the solution $\overrightarrow{v_i}$ to $z_i = \{+1, -1\}$ keeping $\sum_i z_i \approx 0$ *(Approach based on Grothendiek's inequality)*
3. Round z_i to $x_i \in \{0,1\}$

$$\max \sum_{j \in [M]} \| \frac{1}{d} \sum_{i \in \Gamma(j)} \vec{v}_i \|_2^2$$

$$\text{Subject to} \quad \langle \vec{v}_i, \vec{v}_i \rangle \leq 1$$

$$\sum_{i=1}^{n} \vec{v}_i = \vec{0}$$

Disperser Problem (Hardness)

- Given a random bipartite graph, approximate the size of subset in $[N]$ required to hit at least 0.01 fraction of vertices in $[M]$ as its neighbors.
- Hardness related to the Small-Set Expansion (SSE) Hypothesis:

CONJECTURE 1.1. *(Small-Set Expansion Hypothesis [37]) For every constant $\eta > 0$, there exists a small $\delta > 0$ such that given a graph $H = (V, E)$ it is NP-hard to distinguish whether:*

1. *There exists a vertex set S of size $\delta|V|$ such that the edge expansion of S is at most η.*
2. *Every vertex sets S of size $\delta|V|$ has edge expansion at least $1 - \eta$.*

From *Raghavendra and Steurer. Graph Expansion and the Unique Games Conjecture. STOC '10*

Small-Set Expansion Hardness

Theorem 1.5.

For any small constant δ and any constant $\Delta > 1 + \delta$, for appropriately small ρ and large D, it is SSE-hard to distinguish

1. There exists a ρN subset of $[N]$ with at most $(1 - \delta) \cdot \rho N$ neighbors
2. Every ρN subset of $[N]$ has at least $\Delta \cdot \rho N$ neighbors

Conclusion

1. Glossed over the details
2. Several approximation algorithms not mentioned here

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