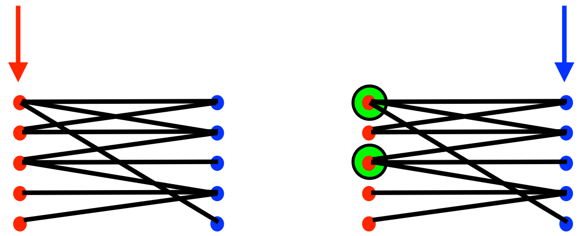
Minimize the k-Union

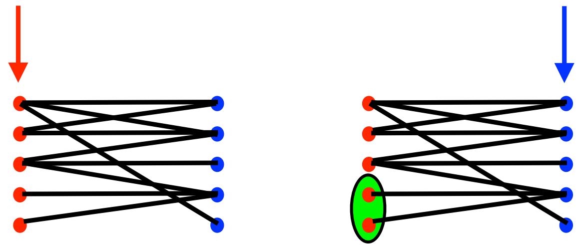
Pick exactly k with largest neighborhood



k=2

- Max-k-Cover
- Approximation?
  (greedy: 1-1/e & tight)

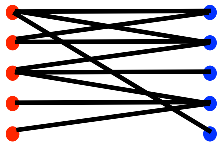
Pick exactly k with smallest neighborhood



k=2

- Min-k-Union
- Approximation-Tightness?

Minimize the k-Union

Results:  L,R sides: n nodes on the left
(not necessarily equal left/right sides)

- Upper bounds: $O(n^{1/4})$
  Transfer technology from D-k-S
  ...not an easy task due to the asymmetry: S + all its neighbors.

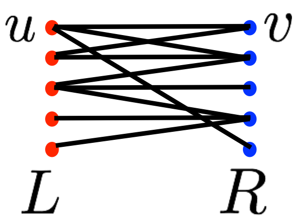
- Lower Bounds:
  - conditional: $\Omega(n^{1/4})$
    Based on Planted vs Random Conjecture...
  - unconditional:

TODAY

1)natural SDP-Integrality gap: $\tilde{\Omega}(\sqrt{n})$
2)Sherali-Adams LP-Integrality gap: $\tilde{\Omega}(n^{1/4})$
for superconstant rounds. ($r \approx \log n / \log\log n$)

Minimize the k-Union

Before SDP, let's see LP:



$$minimize \sum_{v \in R} x_v$$

$$subject\ to: \sum_{u \in L} x_u \geq k$$

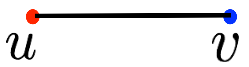
$$x_v \geq x_u \quad (u, v) \in E$$

Now the SDP: $u \in \mathbb{R}^{|L|+|R|}$

$$minimize \sum_{v \in R} ||v||^2$$

$$subject\ to: \sum_{u \in L} ||u||^2 = k$$

$$w_1 \cdot w_2 \geq 0, \forall w_1, w_2 \in L \cup R$$



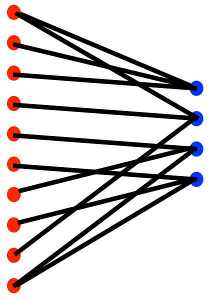
$$u \cdot v = ||u||^2, \forall (u, v) \in E$$

Integrality gap instance?

Minimize the k-Union

Integrality gap instance?

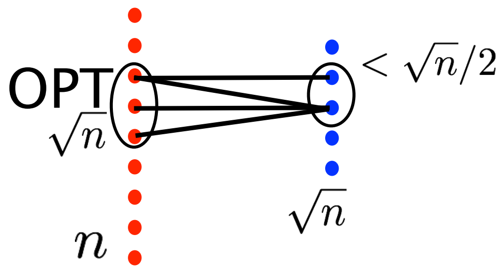
- Same construction for the SDP and for SA
- Bad instances: Random Bipartite Graphs



$$|L| = n$$
$$|R| = \sqrt{n}$$
$$k = \sqrt{n}$$
$$p = \frac{c\log n}{\sqrt{n}}$$

- **Lemma:** integral OPT $\geq \sqrt{n}/2$ w.h.p.

Proof by picture:



$$Pr(no\ edge) \leq (1-p)^{\sqrt{n}\frac{\sqrt{n}}{2}} \leq e^{-\frac{pn}{2}}$$

$$Union\ Bound: \leq n^{\sqrt{n}} \cdot \sqrt{n}^{\sqrt{n}/2} \leq n^{\frac{3\sqrt{n}}{2}}$$

Minimize the k-Union

- **Lemma:** integral OPT $\geq \sqrt{n}/2$ w.h.p. ✓
- **Lemma:** For SDP, exhibit solution: SDP$\leq 4c\log^2 n$ ✓

$p = \frac{c\log n}{\sqrt{n}}$

SDP-Integrality gap: $\tilde{\Omega}(\sqrt{n})$ ✓

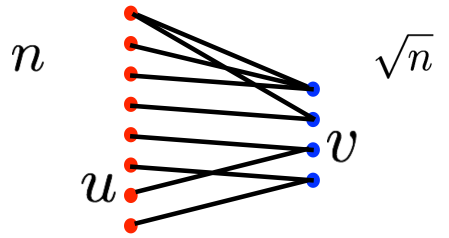
$u \in \mathbb{R}^{|L|+|R|}$

$$minimize \sum_{v\in R} ||v||^2 \checkmark$$

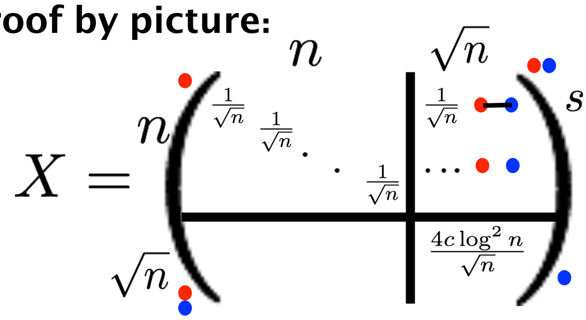
$$subject\ to: \sum_{u\in L} ||u||^2 = k = \sqrt{n} \checkmark$$

$$w_1 \cdot w_2 \geq 0, \forall w_1, w_2 \in L \cup R \checkmark$$

$$u \cdot v = ||u||^2, \forall (u,v) \in E \checkmark$$



Proof by picture:



symmetric p.s.d.

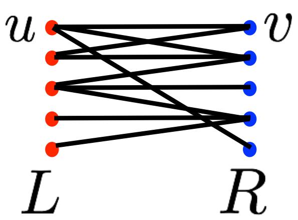
$u_1 \cdot u_2$: top-left

$v_1 \cdot v_2$: bottom-right

$u \cdot v$: other two

Minimize the k-Union

Again the LP:



$$\text{minimize} \sum_{v \in R} x_v$$

$$\text{subject to}: \sum_{u \in L} x_u \geq k$$

$$x_v \geq x_u \quad (u,v) \in E$$

- **Lemma:** OPT $\geq \sqrt{n}/2$
- **Lemma:** SA $\leq n^{1/4}$

↓

SA-Integrality gap: $\tilde{\Omega}(n^{1/4})$

Sherali-Adams:

$$x_S, \forall S \subseteq L \cup R, |S| \leq r$$

$$\text{minimize} \sum_{v \in R} x_{\{v\}}$$

$$\sum_{u \in L} x_{S \cup \{u\},T} \geq k x_{S,T}$$

$$x_{S \cup \{v\},T} \geq x_{S \cup \{u\},T} \quad (u,v) \in E$$

$$x_\varnothing = 1$$

x_S : breaks into three parts

Thanks!