CS369H: Hierarchies of Integer Programming Relaxations Spring 2016-2017

List of Exercises

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This is a running list of problems posed in class. At the end of the course you should submit at least half-of them. The problems are numbered by the lecture number and then by the exercise index. For instance, *Exercise 3.2* would refer to the second exercise given during the third lecture. Often the exercises arise in the context of the lecture where the appropriate definitions are given.

Exercise 1.1. *Show that the constraint* $\{E_x \text{is a valid tour}\}$ *can be expressed using linear constraints for* $x \in \{0,1\}^n$.

Exercise 1.2. *Given a weak separation oracle for a convex cone* K*, show that one can construct a weak separation oracle for* $N(K)$ *and* $N_+(K)$.

Exercise 1.3. *Show that the constraints in (14) are necessary.*

Exercise 2.1. *Show that the constraint* $M_t(y) \succeq 0$ *is satisfied by integral solutions* $y \in \{0,1\}^{2^n}$ *where* $y_I(x) := \prod_{i \in I} x_i$ *for all* $I \subseteq [n]$ *and* $x \in \{0,1\}^n$.

Exercise 3.1. *For Sherali Adams with* 2 *rounds, how high can* $\sum_{i=1}^{n} y_i$ *be for valid solutions* $y \in \mathrm{SA}_2(LP)$.

Exercise 3.2. *Show that* $\sum_i ||v_i||^2 \leq 1$ *when there exists a unit vector* v_0 *such that* $v_i \cdot v_0 = \|v_i\|^2$ *for all* $i \in [n]$ *and* $v_i \perp v_j$ *for all* $i \neq j \in [n]$.

Exercise 3.3. *Show that if in the basic LP we used the weaker constraints:*

$$\sum_{t \leq t'-1} x_{it} \geq x_{jt'},\ \forall i \prec j \tag{1}$$

then for any solution $y \in \mathrm{Las}_t(K)$ *for* $t \geq 3$ *it would be true that:*

$$\sum_{t \leq t'-1} x_{it} \geq \sum_{t \leq t'} x_{jt'},\ \forall i \prec j \tag{2}$$

Hint: use the decomposition theorem an an appropriately chosen set S.

Exercise 4.1. *For* $u\mathbb{R}^d$ *let* $u^{\otimes k} = u \otimes \ldots \otimes u$ *denote the* k*-tensor product of* u*. Show that for all* $u, v \in \mathbb{R}^d$*, it holds that* $\langle u^{\otimes k}, v^{\otimes k} \rangle = (\langle u, v \rangle)^k$.

Exercise 4.2. *For a graph* $G = (V, E)$ *consider its Laplacian* $L_G = \sum_{(i,j)} (e_i - e_j)(e_i - e_j)^T$*. Show that* $\lambda_{max}(L_G)\frac{n}{2} - f_G$ *has degree-2 sos certificate where* f_G *is the max cut polynomial.*

Exercise 4.3. $\forall f : \{0,1\}^n \to \mathbb{R}$ *with degree at most* d *for even* $d \in \mathbb{N}$ *there exists* $M \in \mathbb{R}_{\geq 0}$ *such that* $M - f$ *has degree-*d *sos certificate. Also* M *can be chosen* $n^{O(d)}$ *times the largest coefficient of* f *in the monomial basis.*

Exercise 4.4. *If $\mu : \{0,1\}^n \to \mathbb{R}$ has degree $> \ell$ what is the projection of μ onto U_ℓ? where U_ℓ denotes the linear span of degree-ℓ multilinear polynomials.*

Exercise 4.5. *Show that if μ is a degree-$2n$ pseudo-distribution, then $\mu(x) \geq 0$ for all $x \in \{0,1\}^n$.*

Exercise 4.6. *The following two statements are equivalent:*

1. *μ is a pseudo-distribution.*
2. *$\tilde{E}_\mu 1 = 1$ and $\tilde{E}_\mu \left\{ (1,x)^{\otimes d/2} \left[(1,x)^{\otimes d/2} \right]^\top \right\} \succeq 0$.*

Exercise 4.7. *For all $d \geq 0$ and for any pseudo-distribution μ of degree d, there exists another pseudo-distribution μ' with the same pseudo-moments up to degree d and $|\mu'(x)| \leq 2^{-n} \sum_{d'=0}^{d} \binom{n}{d'}$.*

Exercise 4.8. *For degree d pseudo-distributions over $\{0,1\}^n$ there exists a separation algorithm with running time $n^{O(d)}$.*

Exercise 4.9. *Show that for every $d \in \mathbb{N}$, the following set of pseudo moments admits a separation algorithm with running time $n^{O(d)}$,*

$$\mathcal{M}_d = \left\{ \tilde{\mathbb{E}}_\mu (1,x)^{\otimes d} \middle| \mu \text{ is deg-}d \text{ pseudo distribution over } \{0,1\}^n \right\}.$$

Exercise 5.1. *Prove that Cheeger's inequality $\Rightarrow$ degree 2 SoS certificate for $f_G(x) - \frac{1}{2}\phi^2(G) \cdot \frac{d}{n}|x|(n-|x|)$.*
Hint*: Note the quadratic form of Normalized laplacian can be represented in terms of $f_G(x)$.*

$$\left\langle x, L_G x \right\rangle = \frac{1}{d} f_G(x)$$

Exercise 7.1. *Let v be the max eigenvector of M. Show that w.h.p, $\max_i \langle a_i, v\rangle^2 \geq 1 - o(1)$ and every a_i is still maximal w.p. $n^{-O(1)}$.*

Exercise 7.2. *Suppose we instead defined $T = \sum_{i=1}^{r} a_i^{\otimes 3} + R$ where entries of R are IID Gaussian. How large can R be for the problem to still be well defined? Explain why Jenrich's algorithm will fail.*

Exercise 7.3. *Show that $\|T\|_{sos_k}$ is a norm.*

Exercise 7.4. *Show that the 2-norm of a k-tensor (that is, its 2-norm as a large vector in $\mathbb{R}^{n^k}$) is an upper bound on its sos_d norm, for any $d \geq k$.*

Exercise 7.5. *Show that if k is even, the norm $\|T\|_{op}$ given by unfolding T to a $n^{k/2} \times n^{k/2}$ matrix and measuring its spectral norm satisfies $\|T\|_{op} \geq \|T\|_{sos_k}$.*

Exercise 7.6. *What is the analogue of the sos norm for matrices? Prove that they collapse to $\|M\|_{inj}$.*

Exercise 7.7. *Show that $\left(\tilde{\mathbb{E}} \sum_{i=1}^{r} \langle a_i, x\rangle^2 \right)^{1/2} \leq 1$.*

Exercise 7.8. *Show that the optimization problem:*

$$\begin{aligned} \arg\max \quad & \tilde{\mathbb{E}}\langle T, x^{\otimes 3}\rangle && (3)\\ s.t \quad & deg\tilde{\mathbb{E}} = 6 \text{ satisfies } \{\|x\|^2 = 1\} && (4)\\ & \|\tilde{\mathbb{E}}xx^\top\|_{op} \leq \frac{1}{r} && (5)\\ & \|\tilde{\mathbb{E}}(x\otimes x)(x\otimes x)^\top\|_{op} \leq \tfrac{1}{r} && (6) \end{aligned}$$

is a convex program.

Exercise 7.9. *Show that* $\|\sum_{i\leq n} M_{ij}M_{ij}^\top\|^{1/2} \leq \frac{1}{r}$.

Exercise 8.1. *Provide an argument for why the information theoretic threshold should grow as* $n \geq \Omega(\frac{k\log p}{\lambda^2})$ *(where you can think of* λ *as a small constant and ignore it) and why does it have to grow with the* $\log p$ *of the dimension?*

Exercise 8.2. *Show that the way* x^* *is defined* ($b_t^i \oplus b_t^j \oplus b_t^k = x_i^* \oplus x_j^* \oplus x_k^* \oplus 1$)*, it will be a satisfying assignment.*

Exercise 8.3. *Prove the expansion property for random 3-CSPs.*

Exercise 8.4. *Prove that the last equality* $\sum_i \tilde{\mathbb{E}}(q_i^2) + \sum_{i\neq j}\tilde{\mathbb{E}}(q_iq_j) = \sum_i \tilde{\mathbb{E}}(q_i^2)$ *is true.*

References